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Mixed convection boundary layer flow over a vertical cylinder with prescribed surface heat flux

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Abstract
The steady mixed convection boundary layer flow along a vertical cylinder with prescribed surface heat flux is investigated in this study. The free stream velocity and the surface heat flux are assumed to vary linearly with the distance from the leading edge. Both the case of the buoyancy forces assisting and opposing the development of the boundary layer are considered. Similarity equations are derived, their solutions being dependent on the mixed convection parameter, the curvature parameter, as well as of the Prandtl number. Dual solutions are found to exist for both buoyancy assisting and opposing flows. It is also found that the boundary layer separation is delayed for a cylinder compared to a flat plate.

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1. Introduction

The similarity solutions for combined forced and free convection (mixed convection) flow and heat transfer about a nonisothermal body subjected to a nonuniform free stream velocity were discussed by Sparrow *et al* [1]. They showed that similarity solutions are found to exist when the free stream velocity and the surface temperature vary as x^m and x^{2m-1}, respectively, where x measures the distance from the leading edge and m is a constant. The parameter controlling the relative importance of the free and forced convection is Gr/Re^n, where Gr is the Grashof number, Re is the Reynolds number and n is a constant, which depends on the flow configuration and the surface heating conditions, and is called the buoyancy or mixed convection parameter. This problem was then extended by Merkin and Mahmood [2] to a prescribed wall heat flux case. They found that similarity solutions are possible if the free stream velocity and the wall heat flux vary like x^m and $x^{(5m-3)/2}$, respectively. Merkin and Mahmood [2] analyzed the solutions in terms of the buoyancy parameter and reported the existence of dual solutions for the buoyancy opposing flow (free stream and buoyancy forces

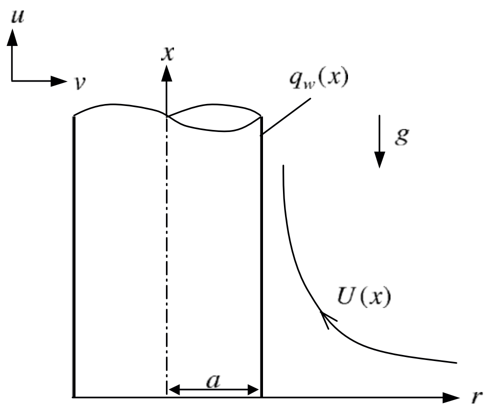

Figure 1. Physical model and the coordinate system (assisting flow).

in the opposite directions). Almost at the same time as [2], Ramachandran *et al* [3] considered a similar problem but for $m = 1$, and they also found that dual solutions exist for the buoyancy opposing flow case. Both prescribed wall temperature and prescribed wall heat flux were considered in [3].

The existence of dual solutions for a certain range of buoyancy parameter was also reported by Wilk and Bramley [4], Devi *et al* [5], Lok *et al* [6] and quite recently by Ishak *et al* [7]. Ingham [8] is probably the first to find dual solutions for the assisting flow case and Ridha [9] for both the opposing and assisting flows. The paper by Ridha [9] showed that dual solutions exist in the opposing flow regime and they continue into that of the assisting flow regime, i.e. when the buoyancy force acts in the same direction as the inertia force.

The present study considers the mixed convection flow and heat transfer along a vertical cylinder with prescribed surface heat flux. The surface heating condition is different from those prescribed wall temperature cases considered by Mahmood and Merkin [10]. As for the flat-plate case mentioned above, we show the existence of dual similarity solutions for both buoyancy assisting and opposing flows. When the curvature parameter is absent, the present problem reduces to the flat-plate case considered by Ramachandran *et al* [3] with which the results can be compared.

2. Problem formulation

Consider a semi-infinite vertical cylinder with radius a placed in a viscous and incompressible fluid of ambient temperature T_∞, as shown in figure 1. It is assumed that the surface of the cylinder is subjected to a variable heat flux $q_w(x)$, and there is a free stream velocity $U(x)$ flowing over the cylinder, and that the buoyancy force can act in the same direction as the flow (assisting flow) or can act in the opposite manner (opposing flow). Under these assumptions, along with the usual boundary layer and Boussinesq approximations, the governing equations are (see [10])

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = U\frac{\mathrm{d}U}{\mathrm{d}x} + \nu\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) + g\beta(T - T_\infty), \tag{2}$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial r}=\alpha\left(\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{3}$$

where x and r are coordinates measured along the surface of the cylinder and in the radial direction, respectively, with u and v being the corresponding velocity components. Further, T is the temperature in the boundary layer, g is the acceleration due to gravity, ν is the kinematic viscosity coefficient, β is the thermal expansion coefficient, and α is the thermal diffusivity. The appropriate boundary conditions are

$$\begin{aligned} &u=0, \qquad v=0, \qquad k\frac{\partial T}{\partial y}=-q_w(x) \qquad \text{at} \quad r=a,\\ &u\to U(x), \qquad T\to T_\infty \qquad \text{as} \quad r\to\infty. \end{aligned} \tag{4}$$

Stewartson [11] showed that the boundary layer flow on a stationary circular cylinder admits similarity solution if the free stream $U(x)$ varies linearly along the axial coordinate. Merkin and Mahmood [2] then showed that similarity solution for the thermal field is possible if the surface heat flux varies linearly in the axial direction, like the free stream. Following Stewartson [11], and Merkin and Mahmood [2], we assume that $U(x)$ and $q_w(x)$ are of the form

$$U(x)=c_1\left(\frac{x}{\ell}\right), \qquad q_w(x)=c_2\left(\frac{x}{\ell}\right), \tag{5}$$

where c_1 and c_2 are constants, and ℓ is a reference length scale. We look for similarity solutions of equations (1)–(3), subject to the boundary conditions (4), by writing (see [2, 10])

$$\begin{aligned} &\eta=\frac{r^2-a^2}{2a}\left(\frac{U}{\nu x}\right)^{1/2}, \qquad \psi=(U\nu x)^{1/2}af(\eta),\\ &T=T_\infty+\frac{q_w}{k}\left(\frac{\nu x}{U}\right)^{1/2}\theta(\eta), \end{aligned} \tag{6}$$

where η is the similarity variable, ψ is the stream function defined as $u=r^{-1}\partial\psi/\partial r$ and $v=-r^{-1}\partial\psi/\partial x$, which identically satisfies equation (1), and k is the thermal conductivity. By defining η in this form, the boundary conditions at $r=a$ reduce to the boundary conditions at $\eta=0$, which is more convenient for numerical computations. From transformation (6), we obtain

$$u=Uf'(\eta) \qquad \text{and} \qquad v=-\frac{a}{r}\left(\frac{\nu c_1}{\ell}\right)^{1/2}f(\eta), \tag{7}$$

where primes denote differentiation with respect to η. Substituting (6) into equations (2) and (3), we obtain the following ordinary differential equations:

$$(1+2\gamma\eta)f'''+2\gamma f''+ff''+1-f'^2+\lambda\theta=0, \tag{8}$$

$$(1+2\gamma\eta)\theta''+2\gamma\theta'+Pr(f\theta'-f'\theta)=0, \tag{9}$$

subject to the boundary conditions (4) which become

$$\begin{aligned} &f(0)=0, \qquad f'(0)=0, \qquad \theta'(0)=-1,\\ &f'(\eta)\to 1, \qquad \theta(\eta)\to 0 \qquad \text{as} \quad \eta\to\infty, \end{aligned} \tag{10}$$

where γ is the curvature parameter, λ is the buoyancy or mixed convection parameter, and Pr is the Prandtl number defined respectively as

$$\gamma=\left(\frac{\nu\ell}{c_1a^2}\right)^{1/2}, \qquad \lambda=\frac{g\beta c_2\nu^{1/2}\ell^{3/2}}{kc_1^{5/2}}=\frac{Gr}{Re^{5/2}}, \qquad Pr=\frac{\nu}{\alpha}, \tag{11}$$

Table 1. Values of $f''(0)$ for different values of Pr when $\lambda = 1$ and $\gamma = 0$ (flat plate).

Pr	Ramachandran *et al* [3]	Present results First solution	Second solution
0.7	1.8339	1.8339	1.2217
1	–	1.7338	1.0218
7	1.4037	1.4037	−0.5357
10	–	1.3712	−0.6057

Table 2. Values of $1/\theta(0)$ for different values of Pr when $\lambda = 1$ and $\gamma = 0$ (flat plate).

Pr	Ramachandran *et al* [3]	Present results First solution	Second solution
0.7	0.7776	0.7776	0.9671
1	–	0.8780	0.7768
7	1.6912	1.6912	−1.5766
10	–	1.9067	−1.9913

with $Gr = g\beta c_2 \ell^4/(k\nu^2)$ and $Re = c_1\ell/\nu$ being the Grashof and Reynolds numbers, respectively. We note that $\lambda > 0$ corresponds to assisting flow (free stream and buoyancy forces in the same direction) and that $\lambda < 0$ corresponds to opposing flow (free stream and buoyancy forces in the opposite directions).

The main physical quantities of interest are the values of $f''(0)$, being a measure of the skin friction, and the non-dimensional surface temperature $\theta(0)$. Our main aim is to find how the values of $f''(0)$ and $\theta(0)$ vary in terms of the parameters γ and λ.

3. Results and discussion

The system of equations (8)–(10) has been solved numerically for some values of the buoyancy parameter λ and curvature parameter γ, while the Prandtl number Pr is fixed to be unity ($Pr = 1$), except for comparisons with previously reported cases. We expect our findings to be qualitatively similar for other values of Pr of $O(1)$. The nonlinear ordinary differential equations (8)–(10) have been solved by two different methods, namely the Keller-box method, described in [12], and the Runge–Kutta method with the shooting technique, described in [13]. For the Keller-box method, the first and the second solutions are obtained by setting different values of η_∞ for the same values of parameters, while for the shooting method, they are obtained by setting different initial guesses for the values of $f''(0)$ and $\theta(0)$, where all profiles satisfy the boundary conditions but with different shapes. The results obtained by both methods are in excellent agreement. Comparisons for the values of $f''(0)$ and $1/\theta(0)$ with those reported by Ramachandran *et al* [3] for the flat-plate case are also made, and they are found to be in favorable agreement, as presented in tables 1 and 2.

Figures 2 and 3 present the velocity and temperature profiles, respectively, for $\lambda = 1$ (assisting flow), which show that the far-field boundary conditions are satisfied, and thus support the validity of the numerical results obtained. The velocity and temperature profiles of valid solutions approach the ambient fluid conditions asymptotically [14]. These figures show that there exist two different profiles for a particular value of λ, where both of them satisfy

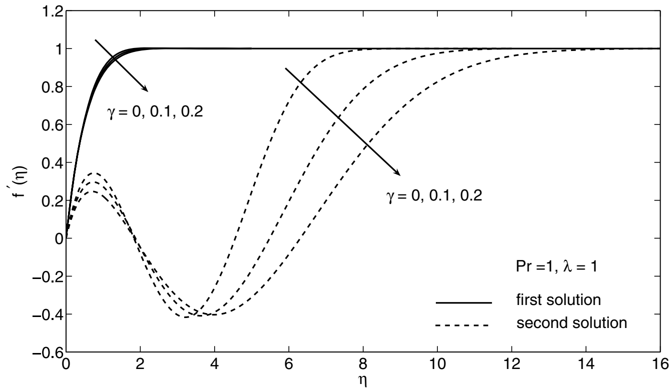

Figure 2. Velocity profiles $f'(\eta)$ for different values of γ when $Pr = 1$ and $\lambda = 1$.

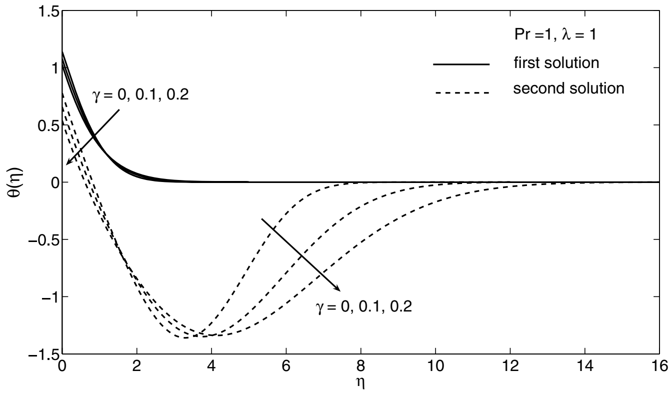

Figure 3. Temperature profiles $\theta(\eta)$ for different values of γ when $Pr = 1$ and $\lambda = 1$.

the far-field boundary conditions. As in similar physical situation, we postulate that the first solutions are stable, whereas the second solutions are not. This postulate can be verified by performing a stability analysis, but this is beyond the scope of the present paper. The second solutions have regions of reversed flow (see figure 2) and this would seem to invalidate them as possible asymptotic solutions to which a fully forward flow developing near the leading edge could evolve. Also, the forced convection limit ($\lambda = 0$) is on the first solution and we expect this solution to be stable, as it is the only solution for this case. Moreover, figure 3 shows that there are regions within the thermal boundary layer where $\theta(\eta) < 0$. This result seems to contradict the second law of thermodynamics.

In an experimental work on turbulent boundary layer under strong adverse pressure gradient, Spangenberg *et al* [15] have found the dual solutions, depending on the manner in which the pressure gradient is applied. Another example of non-unique flow is reported by Aidun *et al* [16] where they have observed experimentally that the primary steady-state flow in a through-flow lid-driven cavity was non-unique, and only one of the multiple steady-state patterns can stabilize in the cavity (see [17, 18]). The first solutions satisfy $f'(\eta) \geqslant 0$ and $\theta(\eta) \geqslant 0$ for all values of η, while the second solutions are characterized by the existence of flow reversal and negative values of $\theta(\eta)$ within the boundary layer (see figures 2 and 3). We

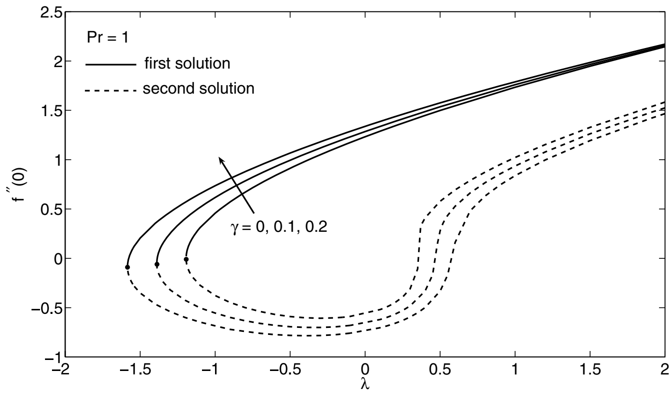

Figure 4. Variation of the skin friction coefficient $f''(0)$ with λ when $Pr = 1$.

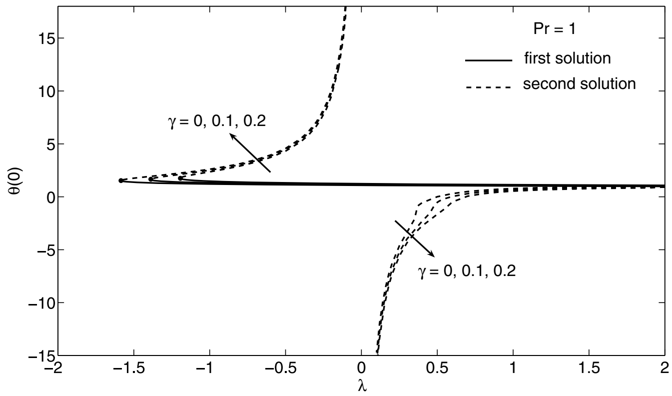

Figure 5. Variation of the surface temperature $\theta(0)$ with λ when $Pr = 1$.

note that the existence of reversed flow in the second solutions has been reported by Ridha [9, 18] and very recently by Ishak *et al* [19, 20].

The variations of the skin friction coefficient $f''(0)$ and the surface temperature $\theta(0)$ with the buoyancy parameter λ for $\gamma = 0, 0.1$ and 0.2 are shown in figures 4 and 5, respectively, both for $Pr = 1$. These figures show that it is possible to obtain dual solutions of the similarity equations (8)–(10) also for the assisting flow ($\lambda > 0$), apart of those for the opposing flow ($\lambda < 0$), that have been typically reported in the literature (see [2, 3, 4, 7]). For $\lambda > 0$, there is a favorable pressure gradient due to the buoyancy forces, which results in the flow being accelerated and consequently there is a larger skin friction coefficient than in the non-buoyant case ($\lambda = 0$) or the opposing flow case ($\lambda < 0$). For negative values of λ, there is a critical value λ_c, with two branches of solutions for $\lambda > \lambda_c$, a saddle-node bifurcation at $\lambda = \lambda_c$ and no solutions for $\lambda < \lambda_c$. These values of λ_c are presented in table 3.

The boundary layer approximation breaks down at $\lambda = \lambda_c$; thus we are unable to obtain further results for $\lambda < \lambda_c$. Beyond this value, the boundary layer has separated from the surface. It is evident from figures 4 and 5 that $|\lambda_c|$ increases with an increase in the curvature parameter γ. The range of λ for which the solution exists is larger for $\gamma > 0$ (cylinder) compared to $\gamma = 0$ (flat plate). Thus, this demonstrates that a cylinder increases the range

Table 3. Values of λ_c for different values of γ when $Pr = 1$.

γ	λ_c
0	−1.1973
0.1	−1.3879
0.2	−1.5845

of existence of the similarity solutions to the equations (8)–(10) compared to a flat plate, i.e. the boundary layer separation is delayed for a cylinder. The results shown in figure 5 for the surface temperature $\theta(0)$ demonstrate that, for the second solution, $\theta(0)$ becomes unbounded as $\lambda \to 0^-$ and as $\lambda \to 0^+$.

4. Conclusions

We have studied the similarity solutions for the steady mixed convection flow past a vertical cylinder with prescribed surface heat flux immersed in an incompressible viscous fluid. The transformed nonlinear ordinary differential equations were solved numerically using two different methods, namely the Keller-box method and the Runge–Kutta method with the shooting technique. We discussed the effects of the curvature parameter γ and the buoyancy parameter λ on the fluid flow and heat transfer characteristics. A new feature to emerge from our results is the existence of a reversed flow region, in addition to a dual solution in the assisting flow regime ($\lambda > 0$). In the assisting flow case, solutions could be obtained for all positive values of λ, while in the opposing flow case, the solution terminated with a saddle-node bifurcation at $\lambda = \lambda_c$ ($\lambda_c < 0$). The value of $|\lambda_c|$ increases with an increase in γ, thus the curvature parameter increases the range of existence of the similarity solutions, which in turn delays the boundary layer breakdown. Hence, the boundary layer separation is delayed for a cylinder ($\gamma > 0$) compared to a flat plate ($\gamma = 0$).

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